

# Expressions of algebra elements and transcendental noncommutative calculus

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**Abstract** Ideas from deformation quantization are applied to deform the expression of elements of an algebra. Extending these ideas to certain transcendental elements implies that  $\frac{1}{i\hbar}uv$  in the Weyl algebra is naturally viewed as an indeterminate living in a discrete set  $\mathbb{N} + \frac{1}{2}$  or  $-(\mathbb{N} + \frac{1}{2})$ . This may yield a more mathematical understanding of Dirac's positron theory.

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## 1 Introduction

Quantum theory is treated algebraically by Weyl algebras, derived from differential calculus via the correspondence principle. However, since the algebra is noncommutative, the so-called *ordering problem* appears.

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Orderings are treated in the physics literature of quantum mechanics (cf. [1]) as the rules of association from classical observables to quantum observables, which are supposed to be self-adjoint operators on a Hilbert space. Typical orderings are, the normal (standard) ordering, the anti-normal (anti-standard) ordering, the Weyl ordering, and the Wick ordering in the case of complex variables.

However, from the mathematical viewpoint, it is better to go back to the original understanding of Weyl, which says that orderings are procedures of realization of the Weyl algebra  $W_{\hbar}$ . Since the Weyl algebra is the universal enveloping algebra of the Heisenberg Lie algebra, the Poincaré-Birkhoff-Witt theorem shows that this algebra can be viewed as an algebra defined on a space of polynomials. As we shall show in §1, this indeed gives product formulas on the space of polynomials which produce algebras isomorphic to  $W_{\hbar}$ . This gives the unique way of expressions of elements, and as a result one can treat transcendental elements such as exponential functions, which are necessary to solve differential equations (cf. §2.2).

However, we encounter several anomalous phenomena, such as elements with two different inverses (cf. §4) and elements which must be treated as double valued (cf. [16],[17]).

In this note, we treat the phenomenon which shows that  $\frac{1}{i\hbar}uv$  should be viewed as an indeterminate living in the set  $\mathbb{N} + \frac{1}{2}$  or  $-(\mathbb{N} + \frac{1}{2})$ . We reach this interpretation in two different ways, by analytic continuation of inverses of  $z + \frac{1}{i\hbar}uv$ , and by defining star gamma functions using various ordering expressions.

The main point is that we do not use operator theory, but instead various ordering expressions, under the leading principle that a physical object should be free from ordering expressions (**the ordering free principle**), just as a geometrical object is free of the local coordinate expressions.

Since similar discrete pictures of elements is familiar in quantum observables, treated as a self-adjoint operator, our observation gives for their justification for the operator theoretic formalism of quantum theory.

However, in this note we restrict our ordering expressions to a particular subset to avoid the multi-valued expressions. In some cases, we should be more careful about the convergence of integrals and the continuity of the product, so the detailed computations and the proof of continuity of the products will appear elsewhere.

## 2 $K$ -ordering expressions for algebra elements

We introduce a method to realize the Weyl algebra via a family of expressions. This leads to a transcendental calculus in the Weyl algebra.

### 2.1 Fundamental product formulas and intertwiners

Let  $\mathfrak{S}_{\mathbb{C}}(n)$  and  $\mathfrak{A}_{\mathbb{C}}(n)$  be the spaces of complex symmetric matrices and skew-symmetric matrices respectively, and  $\mathfrak{M}_{\mathbb{C}}(n) = \mathfrak{S}_{\mathbb{C}}(n) \oplus \mathfrak{A}_{\mathbb{C}}(n)$ . For an arbitrary fixed  $n \times n$ -complex matrix  $\Lambda \in \mathfrak{M}_{\mathbb{C}}(n)$ , we define a product  $*_{\Lambda}$  on the space of polynomials  $\mathbb{C}[\mathbf{u}]$  by the formula

$$(2) \quad f *_{\Lambda} g = f e^{\frac{i\hbar}{2} (\sum \overleftarrow{\partial_{u_i}} \Lambda^{ij} \overrightarrow{\partial_{u_j}})} g = \sum_k \frac{(i\hbar)^k}{k! 2^k} \Lambda^{i_1 j_1} \dots \Lambda^{i_k j_k} \partial_{u_{i_1}} \dots \partial_{u_{i_k}} f \partial_{u_{j_1}} \dots \partial_{u_{j_k}} g.$$

It is known and not hard to prove that  $(\mathbb{C}[\mathbf{u}], *_{\Lambda})$  is an associative algebra.

(a) The algebraic structure of  $(\mathbb{C}[\mathbf{u}], *_{\Lambda})$  is determined by the skew-symmetric part of  $\Lambda$  (in fact, by its conjugacy class  $A \rightarrow {}^t G A G$ ).

(b) In particular, if  $\Lambda$  is a symmetric matrix,  $(\mathbb{C}[\mathbf{u}], *_{\Lambda})$  is isomorphic to the usual polynomial algebra.

Set  $\Lambda=K+J$ ,  $K \in \mathfrak{S}_{\mathbb{C}}(n)$ ,  $J \in \mathfrak{A}_{\mathbb{C}}(n)$ . Changing  $K$  for a fixed  $J$  will be called a *deformation* of expression of elements, as the algebra remains in the same isomorphism class.

**Example of computations:**

$$u_i *_\Lambda u_j = u_i u_j + \frac{i\hbar}{2} \Lambda^{ij}, \quad u_i *_\Lambda u_j *_\Lambda u_k = u_i u_j u_k + \frac{i\hbar}{2} (\Lambda^{ij} u_k + \Lambda^{ik} u_j + \Lambda^{jk} u_i).$$

By computing the  $*_\Lambda$ -product using the product formula (2), every element of the algebra has a unique expression as a standard polynomial. We view these expressions of an element of algebra as analogous to the “local coordinate expression” of a function on a manifold. Thus, changing  $K$  corresponds to a local coordinate transformation on a manifold. In this context, we call the product formula (2) the *K-ordering expression* by ignoring the fixed skew part  $J$ . For  $K=0$ ,  $\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & -I_m \\ -I_m & 0 \end{bmatrix}$ , the  $K$ -ordering expression is called respectively the Weyl ordering, the normal ordering and the anti-normal ordering expressions. The intertwiner between a  $K$ -ordering expression and a  $K'$ -ordering expression, which we view as a local coordinate transformation, is given in a concrete form :

**Proposition 2.1** *For symmetric matrices  $K, K' \in \mathfrak{S}_{\mathbb{C}}(n)$ , the intertwiner is given by*

$$(3) \quad I_K^{K'}(f) = \exp\left(\frac{i\hbar}{4} \sum_{i,j} (K'^{ij} - K^{ij}) \partial_{u_i} \partial_{u_j}\right) f \quad (= I_0^{K'} (I_0^K)^{-1}(f)),$$

*giving an isomorphism  $I_K^{K'} : (\mathbb{C}[\mathbf{u}]; *_K) \rightarrow (\mathbb{C}[\mathbf{u}]; *_{K'+J})$  between algebras. Namely, for any  $f, g \in \mathbb{C}[\mathbf{u}]$  :*

$$(4) \quad I_K^{K'}(f *_K g) = I_K^{K'}(f) *_{K'+J} I_K^{K'}(g).$$

In the case  $n=2m$  and  $J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}$ ,  $(\mathbb{C}[\mathbf{u}], *_\Lambda)$  is called the Weyl algebra, with isomorphism class denoted by  $W_{2m}$ . In fact, if  $J$  is non-singular, then  $(\mathbb{C}[\mathbf{u}], *_\Lambda)$  is isomorphic to the Weyl algebra.

## 2.2 The star exponential function $e_*^{t(z+s\frac{1}{i\hbar}u_k)}$

Using the ordering expression of elements of algebra, we can treat elementary transcendental functions. The  $*$ -exponential function  $e_*^{tH}$  is defined as the family  $:e_*^{tH}:_\Lambda$  of solutions of the evolution equations

$$(5) \quad \frac{d}{dt} f_t = H *_\Lambda f_t, \quad f_0 = 1.$$

For instance, for every  $z \in \mathbb{C}$ , we have

$$(6) \quad :e_*^{z+s\frac{1}{i\hbar}u_k}:_\Lambda = e^z :e_*^{s\frac{1}{i\hbar}u_k}:_\Lambda = e^z e^{s^2 \frac{1}{4i\hbar} K^{kk}} e^{s\frac{1}{i\hbar}u_k}.$$

When we fix the skew part  $J$  of  $\Lambda$ , we often abbreviate the notation to  $: :_K, *_K$  for  $: :_{K+J}, *_{K+J}$  respectively.

Since the exponential law

$$:e_*^{(z+w)+(s+t)\frac{1}{i\hbar}u_k}:_K = :e_*^{z+s\frac{1}{i\hbar}u_k}:_K *_K :e_*^{w+t\frac{1}{i\hbar}u_k}:_K$$

holds for every  $K$ , it is better to write

$$e_*^{(z+w)+(s+t)\frac{1}{i\hbar}u_k} = e_*^{z+s\frac{1}{i\hbar}u_k} *_K e_*^{w+t\frac{1}{i\hbar}u_k}$$

by viewing  $:e_*^{z+s\frac{1}{i\hbar}u_k}:_K$  as the  $K$ -ordering expression of the (ordering free) exponential element  $e_*^{z+s\frac{1}{i\hbar}u_k}$ . Under this convention, one may write for instance  $:u_i*u_j:_K = u_i u_j + \frac{i\hbar}{2}(K+J)^{ij}$ .

We remark that even for the simplest exponential function  $e_*^{s\frac{1}{i\hbar}u_k}$ , formula (6) gives the following (cf. [13]).

**Proposition 2.2** *If  $\text{Im } K^{kk} < 0$ , then the  $K$ -ordering expression of  $\sum_{n \in \mathbb{Z}} e_*^{2n\frac{1}{i\hbar}u_k}$  converges, and  $:\sum_{n \in \mathbb{Z}} e_*^{2n\frac{1}{i\hbar}u_k}:_K$  is precisely the Jacobi theta function  $\theta_3(\frac{1}{i\hbar}u_k)$ .*

This shows that deformations of expressions of a fixed algebraic system are interesting in their own right (cf. [5]). However, it should be remarked that  $\sum_{n=0}^{\infty} e_*^{2n\frac{1}{i\hbar}u_k}$ , and  $-\sum_{n=-\infty}^{-1} e_*^{2n\frac{1}{i\hbar}u_k}$  each converge to inverses of  $1 - e_*^{\frac{1}{i\hbar}u_k}$ . This leads to a breakdown of associativity. Such phenomena occur very often in a transcendently extended algebraic system.

If  $\text{Im } K^{kk} < 0$ , then the  $K$ -ordering expression of the integral  $\int_{\mathbb{R}} e_*^{t\frac{1}{i\hbar}u_k} dt$  converges, and

$$(7) \quad e_*^{z\frac{1}{i\hbar}u_k} * \int_{\mathbb{R}} e_*^{t\frac{1}{i\hbar}u_k} dt = \int_{\mathbb{R}} e_*^{t\frac{1}{i\hbar}u_k} dt, \quad \forall z \in \mathbb{C}.$$

However, we have shown in [18] that  $\int_{\mathbb{R}} e_*^{t\frac{1}{i\hbar}u_k} dt$  is double valued.

### 3 Star exponential functions of quadratic forms

In this note we mainly deal with the Weyl algebra  $W_2$  over  $\mathbb{C}$ . Putting  $u_1 = u, u_2 = v$ , we have the commutation relation  $[u, v] = -i\hbar$ , where  $[u, v] = uv - v*u$ . The product formula (2) with  $\Lambda = K + J, J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  realizes  $W_2$ .

In what follows, we use the following notations:

$$(8) \quad u*v = v*u - i\hbar, \quad uv = \frac{1}{2}(u*v + v*u), \quad v*u = uv + \frac{1}{2}i\hbar.$$

Let  $K = \begin{bmatrix} 0 & \kappa \\ \kappa & 0 \end{bmatrix}$ . The product  $*_{\kappa}$  and the ordering expression  $: :_{\kappa}$  stand for  $*_K$  and  $: :_K$ , respectively. Namely,  $*_0$  and  $*_1$  correspond to the Moyal product and the standard product. We also denote the intertwiner from the  $*_{\kappa}$ -product to the  $*_{\kappa'}$  product by  $I_{\kappa}^{\kappa'}$ .

Let  $\text{Hol}(\mathbb{C}^2)$  be the set of holomorphic functions  $f(u, v)$  on the complex 2-plane  $\mathbb{C}^2$  endowed with the topology of uniform convergence on compact subsets.  $\text{Hol}(\mathbb{C}^2)$  is viewed as a Fréchet space.

The following fundamental lemma follows easily from the product formula (2).

**Lemma 3.1** *For every polynomial  $p(u, v)$ , left multiplication  $p(u, v)*$  (resp. right multiplication  $*p(u, v)$ ) is a continuous linear mapping of  $\text{Hol}(\mathbb{C}^2)$  into itself.*

#### 3.1 The star exponential function $e_*^{t(z+\frac{1}{i\hbar}uv)}$

If  $f_t = h(uv)$  in (5), then  $I_{\kappa}^{\kappa'}(h(uv))$  is also a function of  $uv$ . From here on, we mainly concern with functions of  $uv$  alone. We set  $\frac{2}{i\hbar}uv = \mathbf{u}A\mathbf{u}$ , where  $\mathbf{u} = (u, v)$  and  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . The intertwiner  $I_{\kappa}^{\kappa'}$  is given as follows:

$$(9) \quad I_{\kappa}^{\kappa'}(ge^{t\frac{2}{i\hbar}uv}) = g \frac{1}{1-t(\kappa'-\kappa)} e^{\frac{t}{1-t(\kappa'-\kappa)}\frac{2}{i\hbar}uv}$$

Solving the evolution equation (5) for the exponential function, we see that  $e_*^{t\frac{1}{i\hbar}2uv}$  is given by

$$(10) \quad :e_*^{t\frac{1}{i\hbar}2uv}:_0 = \frac{1}{\cosh t} e^{\frac{1}{i\hbar}2uv \tanh t}$$

in the Weyl ordering expression (cf. [16]), and by

$$(11) \quad :e_*^{t\frac{1}{i\hbar}2uv}:_I = e^t e^{\frac{1}{i\hbar}(e^{2t}-1)uv}$$

in the normal ordering expression (cf.[14]).

Since  $:e_*^{t\frac{2}{i\hbar}uv}:_\kappa = I_0^\kappa(\frac{1}{\cosh t} e^{\frac{1}{i\hbar}2uv \tanh t})$ , we see that

$$(12) \quad :e_*^{t\frac{1}{i\hbar}2uv}:_\kappa = \frac{2}{(1-\kappa)e^t + (1+\kappa)e^{-t}} \exp\left(\frac{e^t - e^{-t}}{(1-\kappa)e^t + (1+\kappa)e^{-t}} \frac{1}{i\hbar}2uv\right).$$

Let  $K = \begin{bmatrix} 0 & \kappa \\ \kappa & \tau \end{bmatrix}$ . The product  $*_{(\kappa,\tau)}$  and the ordering expression  $: :_{(\kappa,\tau)}$  stand for  $*_\kappa$  and  $: :_\kappa$ , respectively.

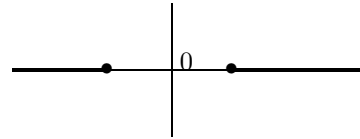
It is not hard to obtain the  $(\kappa, \tau)$ -ordering expression:

$$(13) \quad :e_*^{t\frac{1}{i\hbar}2uv}:_{(\kappa,\tau)} = \frac{2}{\Delta} \exp\left(\left(\frac{e^t - e^{-t}}{\Delta}\right)^2 \tau \frac{1}{i\hbar}u^2 + \frac{e^t - e^{-t}}{\Delta} \frac{1}{i\hbar}2uv\right), \quad \Delta = (e^t + e^{-t}) - \kappa(e^t - e^{-t}),$$

where  $\Delta = (e^t + e^{-t}) - \kappa(e^t - e^{-t})$ . The general ordering expression is a little more complicated involving the square root in the amplitude.

Note that  $(1-\kappa)e^t + (1+\kappa)e^{-t} = 0$  if and only if  $e^{2t} = \frac{\kappa+1}{\kappa-1}$ . Hence,  $:e_*^{t\frac{1}{i\hbar}2uv}:_{(\kappa,\tau)}$  has a singular point at  $2t = \log \frac{\kappa+1}{\kappa-1} + 2\pi i\mathbb{Z}$ . However, if  $\kappa = \pm 1$ , then  $:e_*^{t\frac{1}{i\hbar}2uv}:_{(\pm 1,\tau)}$  are entire functions with respect to  $t$ . In general we have the following:

**Lemma 3.2** *If  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ , then the  $(\kappa, \tau)$ -ordering expression  $:e_*^{t\frac{2}{i\hbar}uv}:_{(\kappa,\tau)}$  is real analytic and rapidly decreasing with respect to  $t \in \mathbb{R}$ .*



Formula (13) gives also the following:

**Proposition 3.1** *Suppose  $\kappa \neq 0$ ,  $z \in \mathbb{C}$ . Then the  $(\kappa, \tau)$ -ordering expression  $:\sin_* \pi(z + \frac{1}{i\hbar}uv):_{(\kappa,\tau)}$  is holomorphic in  $(z, uv)$ , and vanishes on  $z \in \mathbb{Z} + \frac{1}{2}$ .*

**Proof** By (13),  $:e_*^{\pi i \frac{1}{i\hbar}2uv}:_{(\kappa,\tau)} + 1 = 0$ . Although the Weyl ordering expression (the case  $\kappa=0$ ) of  $e_*^{\pm \pi i \frac{1}{i\hbar}uv}$  diverges by (10), other ordering expressions exist, e.g. (in normal ordering)

$$:e_*^{\pi i \frac{1}{i\hbar}uv}:_1 = i e^{-\frac{1}{i\hbar}2uv}, \quad :e_*^{-\pi i \frac{1}{i\hbar}uv}:_1 = -i e^{-\frac{1}{i\hbar}2uv}.$$

Thus, we have

$$0 = e_*^{-\pi i \frac{1}{i\hbar}uv} * (e_*^{\pi i \frac{1}{i\hbar}2uv} + 1) = e_*^{\pi i \frac{1}{i\hbar}uv} + e_*^{-\pi i \frac{1}{i\hbar}uv} = 2 \cos_*\left(\pi \frac{1}{i\hbar}uv\right).$$

The desired result follows from the exponential law.  $\square$

**Lemma 3.3** *If  $\sin_* \pi(z + \frac{1}{i\hbar}uv) * f(uv)$  is defined on some domain containing  $z = \frac{1}{2}$ , then  $\sin_* \pi(\frac{1}{2} + \frac{1}{i\hbar}uv) * f(uv) = 0$ .*

These observations lead to viewing  $\frac{1}{2} + \frac{1}{i\hbar}uv$  is an indeterminate in the set of integers  $\mathbb{Z}$ , that is,  $\frac{1}{i\hbar}v*u$  behaves as if it were an indeterminate in  $\mathbb{Z}$ . However, we have to keep in mind the following remark:

**Remark 1** There are two definitions of the product  $e_*^{z\frac{1}{i\hbar}uv} * f(u, v)$ . The first is to define as the real analytic solution of

$$\frac{d}{dt}f_t = \frac{1}{i\hbar}uv * f_t, \quad f_0 = f(u, v),$$

if a real analytic solution exist. The second is to define

$$e_*^{z\frac{1}{i\hbar}uv} * f(u, v) = \lim_{n \rightarrow \infty} e_*^{z\frac{1}{i\hbar}uv} * f_n(u, v), \quad \text{if } f(u, v) = \lim_n f_n(u, v),$$

where  $f_n$  are polynomials. These two definitions do not agree in general, since the multiplication  $e_*^{z\frac{1}{i\hbar}uv} *$  is not a continuous linear mapping of  $Hol(\mathbb{C}^2)$  into itself (cf.(17)).

### 3.2 Several estimates

We have already known that  $:e_*^{t\frac{1}{i\hbar}uv} :_\kappa \in Hol(\mathbb{C}^2)$  for every fixed  $t$  whenever defined. By (12), we see also that if  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ , then  $:e_*^{t\frac{1}{i\hbar}uv} :_\kappa$  is rapidly decreasing with respect to  $t$ .

In this section, we first show that  $\int_{-\infty}^{\infty} e_*^{t\frac{1}{i\hbar}uv} dt \in Hol(\mathbb{C}^2)$  in the Weyl ordering expression.

The Weyl ordering expression of  $e_*^{t\frac{1}{i\hbar}uv}$  is  $:e_*^{t\frac{1}{i\hbar}uv} :_0 = \frac{1}{\cosh \frac{t}{2}} e^{(\tanh \frac{t}{2})\frac{1}{i\hbar}2uv}$ . Hence

$$: \int_{\mathbb{R}} e_*^{t\frac{1}{i\hbar}uv} dt :_0 = \int_{-\infty}^{\infty} \frac{1}{\cosh \frac{t}{2}} e^{(\tanh \frac{t}{2})\frac{1}{i\hbar}2uv} dt.$$

By setting  $\cos s = \tanh \frac{t}{2}$ ,  $-2 \sin s ds = \sin^2 s dt$ , the integral on the right hand side becomes into

$$2 \int_{-\pi}^0 e^{(\cos s)\frac{1}{i\hbar}2uv} ds = \int_{-\pi}^{\pi} e^{(\cos s)\frac{1}{i\hbar}2uv} ds.$$

By the Hansen-Bessel formula, we have

$$(14) \quad : \int_{-\infty}^{\infty} e_*^{t\frac{1}{i\hbar}uv} dt :_0 = \sqrt{\frac{\pi}{2}} J_0\left(\frac{2}{\hbar}uv\right),$$

where  $J_0$  is Bessel function of eigen value 0.

Since  $g(s) = e^{(\cos s)\frac{1}{i\hbar}2uv}$  is a continuous curve in  $Hol(\mathbb{C}^2)$ , its integral (14) on a compact domain belongs to  $Hol(\mathbb{C}^2)$ .

Applying the intertwiner  $I_0^\kappa$  for (14), we see that  $: \int_{\mathbb{R}} e_*^{t\frac{1}{i\hbar}uv} dt :_\kappa = \int_{-\pi}^{\pi} :e^{(\cos s)\frac{1}{i\hbar}2uv} :_\kappa ds$ . Since

$$:e^{(\cos s)\frac{1}{i\hbar}2uv} :_\kappa = \frac{2}{(1-\kappa)e^{\frac{1}{2}\cos s} + (1+\kappa)e^{\frac{1}{2}\cos s}} \exp\left(\frac{e^{\cos s} - 1}{(1-\kappa)e^{\cos s} + (1+\kappa)} \frac{1}{i\hbar}2uv\right),$$

we have the following:

**Proposition 3.2** *For every  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ , the  $\kappa$ -ordering expression of the integral  $: \int_{-\infty}^{\infty} e_*^{t\frac{1}{i\hbar}uv} dt :_\kappa$  is contained in the space  $Hol(\mathbb{C}^2)$ . Furthermore, integration by parts gives  $\frac{d}{d\theta} \int_{-\infty}^{\infty} e_*^{e^{i\theta}t\frac{1}{i\hbar}uv} e^{i\theta} dt = 0$  whenever defined.*

The  $*$ -delta function is defined by the following integral:

$$\int_{-\infty}^{\infty} e_*^{t\frac{1}{i\hbar}uv} dt = \int_{\mathbb{R}} e_*^{-it\frac{1}{\hbar}uv} dt = \delta_*(\frac{1}{\hbar}uv).$$

Note that  $\cos s = \tanh \frac{t}{2}$  implies  $t = \log \frac{1+\cos s}{1-\cos s}$ . Hence, we have

**Lemma 3.4** *If  $f(t)$  is a continuous function such that  $f(\log \frac{1+\cos s}{1-\cos s})$  is continuous on  $[-\pi, 0]$ , then  $\int_{-\infty}^{\infty} f(t) e_*^{t\frac{1}{i\hbar}uv} dt$  is in  $Hol(\mathbb{C}^2)$  in the  $\kappa$ -ordering expression such that for every  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ .*

Applying Lemma 3.4 to the function  $f(t) = e^{-at}$  ( $a > 0$ ) and  $e^{-e^t}$ , we have:  $\int_{\mathbb{R}} e^{-at} e_*^{t\frac{1}{i\hbar}uv} dt;_{\kappa}$  and  $\int_{\mathbb{R}} e^{-e^t} e_*^{t\frac{1}{i\hbar}uv} dt;_{\kappa}$  are elements of  $Hol(\mathbb{C}^2)$ . We denote the second integral by

$$\int_{\mathbb{R}} e^{-e^t} e_*^{t\frac{1}{i\hbar}uv} dt = \Gamma_*(\frac{1}{i\hbar}uv) \quad (\text{cf. §6}).$$

Since  $v*u = uv + \frac{1}{2}i\hbar$ , (13) also gives the existence of the limit

$$(15) \quad \begin{aligned} \lim_{t \rightarrow \infty} :e_*^{t\frac{1}{i\hbar}2v*u} :_{(\kappa, \tau)} &= \frac{2}{1-\kappa} e^{\frac{1}{i\hbar} \frac{1}{1-\kappa} (2uv + \frac{\tau}{1-\kappa} u^2)}, \\ \lim_{t \rightarrow -\infty} :e_*^{t\frac{1}{i\hbar}2u*v} :_{(\kappa, \tau)} &= \frac{2}{1+\kappa} e^{-\frac{1}{i\hbar} \frac{1}{1+\kappa} (2uv - \frac{\tau}{1+\kappa} u^2)}, \\ \lim_{t \rightarrow -\infty} :e_*^{t\frac{1}{i\hbar}2v*u} :_{(\kappa, \tau)} &= 0, \quad \lim_{t \rightarrow \infty} :e_*^{t\frac{1}{i\hbar}2u*v} :_{(\kappa, \tau)} = 0. \end{aligned}$$

We call

$$\varpi_{00} = \lim_{t \rightarrow -\infty} e_*^{t\frac{1}{i\hbar}2u*v}, \quad \overline{\varpi}_{00} = \lim_{t \rightarrow \infty} e_*^{t\frac{1}{i\hbar}2u*v}$$

**vacuums.** The exponential law gives

$$\varpi_{00} *_0 \varpi_{00} = \varpi_{00}, \quad \overline{\varpi}_{00} *_0 \overline{\varpi}_{00} = \overline{\varpi}_{00}.$$

However, we easily see

**Theorem 3.1** *The product  $\varpi_{00} *_0 \overline{\varpi}_{00}$  diverges in any ordering expression.*

The existence of the limit (15) gives also

$$u*v*\varpi_{00} = 0 = \varpi_{00}*u*v.$$

But the “bumping identity”  $v*f(u*v) = f(v*u)*v$  give the following:

**Lemma 3.5**  $v*\varpi_{00} = 0 = \varpi_{00}*u.$

**Proof** Using the continuity of  $v*$ , we see that  $v*\lim_{t \rightarrow -\infty} e_*^{t\frac{1}{i\hbar}2u*v} = \lim_{t \rightarrow -\infty} v*e_*^{t\frac{1}{i\hbar}2u*v}$ . Hence, the bumping identity (proved by the uniqueness of the real analytic solution for linear differential equations) gives  $\lim_{t \rightarrow -\infty} e_*^{t\frac{1}{i\hbar}2v*u} *v = 0$  by using (15).  $\square$ .

However, we note that associativity is not easily ensured. The following is the simplest condition which ensures associativity for certain calculations:

**Proposition 3.3** *For every polynomial and for every entire function  $f \in Hol(\mathbb{C}^2)$ , the products  $p*f$  and  $f*p$  are defined as elements of  $Hol(\mathbb{C}^2)$ , and associativity  $(f*g)*h = f*(g*h)$  holds whenever two of  $f, g, h$  are polynomials.*

In general  $(f*g)*h=f*(g*h)$  does not hold even if  $g$  is a polynomial.

**Example 1** By Lemma 3.2,  $\frac{1}{i\hbar}uv$  has two different inverses

$$\left(\frac{1}{i\hbar}uv\right)_+^{-1} = \int_{-\infty}^0 e_*^{t\frac{1}{i\hbar}uv} dt, \quad \left(\frac{1}{i\hbar}uv\right)_-^{-1} = - \int_0^{\infty} e_*^{t\frac{1}{i\hbar}uv} dt$$

as elements of  $Hol(\mathbb{C}^2)$ . Hence, we see the failure of associativity :

$$\left(\left(\frac{1}{i\hbar}uv\right)_+^{-1} * \left(\frac{1}{i\hbar}uv\right)\right) * \left(\frac{1}{i\hbar}uv\right)_-^{-1} \neq \left(\frac{1}{i\hbar}uv\right)_+^{-1} * \left(\left(\frac{1}{i\hbar}uv\right) * \left(\frac{1}{i\hbar}uv\right)_-^{-1}\right),$$

and indeed  $\left(\frac{1}{i\hbar}uv\right)_+^{-1} * \left(\frac{1}{i\hbar}uv\right)_-^{-1}$  diverges in any ordering expression. In what follows, we use the notation

$$(16) \quad \delta_*\left(\frac{1}{\hbar}uv\right) = \left(\frac{1}{i\hbar}uv\right)_+^{-1} - \left(\frac{1}{i\hbar}uv\right)_-^{-1}.$$

In spite of theis general failure of associativity, we have another primitive criterion for associativity. We remark that if all terms are considered as formal power series in  $i\hbar$  in the product formula (2), then the product is always defined, and it is easy to show associativity, as it holds for polynomials (cf. [14] for details). Applying these remarks carefully, we give the following:

**Lemma 3.6**  $\varpi_{00}*(u^p*\varpi_{00})=0$ , and  $(\varpi_{00}*v^p)*\varpi_{00}=0$ .

**Proof** By taking the formal power series expansion with respect to  $i\hbar$  for  $e_*^{su*uv}$ , associativity holds, and the following computation is permitted by the bumping identity:

$$e_*^{su*uv} * (u^p * e_*^{tu*uv}) = (e_*^{su*uv} * u^p) * e_*^{tu*uv} = u^p * e_*^{(s+t)u*uv + i\hbar ps}.$$

The right hand side of the above equality is continuous in  $s, t$ . In particular,

$$\lim_{t \rightarrow a} e_*^{su*uv} * (u^p * e_*^{tu*uv}) = e_*^{su*uv} * \lim_{t \rightarrow a} (u^p * e_*^{tu*uv}).$$

Using the bumping identity, we have

$$\begin{aligned} e_*^{su*uv} * (u^p * \lim_{t \rightarrow -\infty} e_*^{tu*uv}) &= e_*^{su*uv} * \lim_{t \rightarrow -\infty} u^p * e_*^{tu*uv} = \lim_{t \rightarrow -\infty} u^p * e_*^{(s+t)u*uv + i\hbar ps} \\ &= u^p * \lim_{t \rightarrow -\infty} e_*^{(s+t)u*uv + i\hbar ps} = u^p e^{i\hbar ps} * \varpi_{00}. \end{aligned}$$

It follows that

$$\varpi_{00}*(u^p*\varpi_{00}) = \lim_{s \rightarrow -\infty} e_*^{s\frac{1}{i\hbar}u*uv} * \left(\lim_{t \rightarrow -\infty} u^p * e_*^{t\frac{1}{i\hbar}u*uv}\right) = \lim_{s \rightarrow -\infty} u^p e^{ps} * \varpi_{00} = 0.$$

Similarly, we also have  $(\varpi_{00}*v^p)*\varpi_{00}=0$ . □

**Lemma 3.7** For every polynomial  $f(u, v) = \sum a_{ij} u^i * v^j$ ,

$$\varpi_{00}*(f(u, v)*\varpi_{00}) = f(0, 0)\varpi_{00} = (\varpi_{00}*f(u, v))*\varpi_{00}.$$

Consequently, associativity holds for  $\varpi_{00}*p(u, v)*\varpi_{00}$  for a polynomial  $p(u, v)$ .

A similar computation gives the following associativity

$$(\varpi_{00}*v^q)*(u^p*\varpi_{00}) = \delta_{p,q} p!(i\hbar)^p = \varpi_{00}*(v^q*u^p*\varpi_{00}) = (\varpi_{00}*v^q*u^p)*\varpi_{00}.$$

Since

$$\varpi_{00}*v^q*u^p*\varpi_{00} = \delta_{p,q} p!(i\hbar)^p \varpi_{00},$$

we have the following:



**Proposition 3.4**  $\frac{1}{\sqrt{p!q!(i\hbar)^{p+q}}}u^p*\varpi_{00}*v^q$  is the  $(p, q)$ -matrix element.

As mentioned in Remark 1 in § 3.1, we have two definitions of  $e_*^{z\frac{1}{i\hbar}uv}*f(u, v)$ . However both definitions give the formula

$$(17) \quad e_*^{z\frac{1}{i\hbar}uv}*\varpi_{00}=e^{-\frac{1}{2}z}*\varpi_{00}.$$

On the other hand, since  $\frac{1}{i\hbar}uv*\delta_*(\frac{1}{i\hbar}uv)=0$ , we must set  $e_*^{t\frac{1}{i\hbar}uv}*\delta_*(\frac{1}{i\hbar}uv)=\delta_*(\frac{1}{i\hbar}uv)$  as the real analytic solution of  $\frac{d}{dt}f_t=\frac{1}{i\hbar}uv*f_t$ .

However, computing

$$\lim_{N \rightarrow \infty} e_*^{t\frac{1}{i\hbar}uv} * \int_{-N}^N e_*^{s\frac{1}{i\hbar}uv} ds = \lim_{N \rightarrow \infty} \int_{-N}^N e_*^{(t+s)\frac{1}{i\hbar}uv} ds$$

gives the following:

$$(18) \quad e_*^{(x+iy)\frac{1}{i\hbar}uv}*\delta_*(\frac{1}{i\hbar}uv)=e_*^{iy\frac{1}{i\hbar}uv}*\delta_*(\frac{1}{i\hbar}uv).$$

Hence (17) is holomorphic with respect to  $z$ , while (18) is only continuous, that is, there is no real analyticity with respect to  $z = x+iy$ .

## 4 Inverses and their analytic continuation

Formula (6) and the exponential law give in particular

$$:e_*^{t(z+\frac{1}{i\hbar}v)}:_{(\kappa,\tau)}=e^{\frac{1}{4i\hbar}t^2\tau}e^{t(z+\frac{1}{i\hbar}v)}.$$

It follows that if  $\text{Im } \tau < 0$ , then  $e^{\frac{1}{4i\hbar}t^2\tau}$  is rapidly decreasing in  $t$  and the integrals

$$(19) \quad : \int_{-\infty}^0 e_*^{t(z+\frac{1}{i\hbar}v)} dt :_{(\kappa,\tau)}, \quad - : \int_0^{\infty} e_*^{t(z+\frac{1}{i\hbar}v)} dt :_{(\kappa,\tau)}.$$

converge. Both integrals are respectively inverses of  $z+\frac{1}{i\hbar}v$ , and are denoted  $(z+\frac{1}{i\hbar}v)_{+*}^{-1}$ ,  $(z+\frac{1}{i\hbar}v)_{-*}^{-1}$ , respectively, with the subscript  $(\kappa, \tau)$  omitted.

**Proposition 4.1** *If  $\text{Im } \tau < 0$ , then the  $(\kappa, \tau)$ -ordering expression of the difference of the two inverses is given by*

$$:(z+\frac{1}{i\hbar}v)_{+*}^{-1}-(z+\frac{1}{i\hbar}v)_{-*}^{-1}:_{(\kappa,\tau)}=\int_{-\infty}^{\infty} e^{\frac{1}{4i\hbar}t^2\tau}e^{t(z+\frac{1}{i\hbar}v)} dt.$$

*This difference is holomorphic in  $z$ .*

Similarly, by formula (10), we have the convergence of the two integrals

$$(20) \quad : \int_{-\infty}^0 e^{tz} e_*^{t\frac{1}{i\hbar}uv} dt :_0 = \int_{-\infty}^0 \frac{e^{\frac{1}{2}tz}}{\cosh \frac{1}{2}t} e^{\frac{1}{i\hbar}2uv \tanh \frac{1}{2}t} dt, \quad \text{Re } z > -\frac{1}{2},$$

$$(21) \quad : - \int_0^{\infty} e^{tz} e_*^{t\frac{1}{i\hbar}uv} dt :_0 = - \int_0^{\infty} \frac{e^{\frac{1}{2}tz}}{\cosh \frac{1}{2}t} e^{\frac{1}{i\hbar}2uv \tanh \frac{1}{2}t} dt, \quad \text{Re } z < \frac{1}{2}.$$

Both (20) and (21) give inverses of  $z+\frac{1}{i\hbar}uv$ . By a similar computation, there are two inverses for every  $(\kappa, \tau)$  such that  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ , which will be denoted by  $(z+\frac{1}{i\hbar}uv)_{+*}^{-1}$ ,  $(z+\frac{1}{i\hbar}uv)_{-*}^{-1}$ .

The following may be viewed as a Sato hyperfunction:

**Proposition 4.2** *If  $-\frac{1}{2} < \operatorname{Re} z < \frac{1}{2}$ , then the difference of the two inverses is given by*

$$(22) \quad (z + \frac{1}{i\hbar}uv)_{+*}^{-1} - (z + \frac{1}{i\hbar}uv)_{-*}^{-1} = \int_{-\infty}^{\infty} e_*^{t(z + \frac{1}{i\hbar}uv)} dt.$$

*Its  $(\kappa, \tau)$ -ordering expression is holomorphic on this strip.*

One can see the right hand side more closely. For  $-\frac{1}{2} < \operatorname{Re} z \leq 0$ , the change of variables  $\tanh \frac{1}{2}t = \cos s$  from forms the right hand side of (22) into

$$2 \int_{-\pi}^0 \left( \frac{1 + \cos s}{1 - \cos s} \right)^z e^{(\cos s) \frac{1}{i\hbar} 2uv} ds.$$

For  $0 \leq \operatorname{Re} z < \frac{1}{2}$  and for  $-\cos s = \tanh \frac{t}{2}$ ,  $2 \sin s ds = \sin^2 s dt$ , the right hand side of (22) transforms into

$$2 \int_0^\pi \left( \frac{1 + \cos s}{1 - \cos s} \right)^{-z} e^{(\cos s) \frac{1}{i\hbar} uv} ds.$$

Hence, Lemma 3.4 gives that  $\int_{-\infty}^{\infty} e_*^{t(z + \frac{1}{i\hbar}uv)} dt$  is an element of  $Hol(\mathbb{C}^2)$ .

On the other hand, note that a change of variables gives

$$((-z) + \frac{1}{i\hbar}uv)_{-*}^{-1} = - \int_0^\infty e_*^{-t(z - \frac{1}{i\hbar}uv)} dt = - \int_{-\infty}^0 e_*^{(z - \frac{1}{i\hbar}uv)} dt.$$

Thus, we see that

$$(23) \quad (z - \frac{1}{i\hbar}uv)_{-*}^{-1} = -((-z) + \frac{1}{i\hbar}uv)_{-*}^{-1}.$$

This is holomorphic on the domain  $\operatorname{Re} z > -\frac{1}{2}$ , which is also the holomorphic domain for  $(z + \frac{1}{i\hbar}uv)_{-*}^{-1}$ .

All of these results are easily proved for the Weyl ordering expression. However, if  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ , then  $:e_*^{t \frac{1}{i\hbar}uv} :_\kappa$  is rapidly decreasing in  $t$ , and the same computation gives the following:

**Proposition 4.3** *For every  $z$  such that  $\operatorname{Re} z > -\frac{1}{2}$ , the two inverses  $(z + \frac{1}{i\hbar}uv)_{+*}^{-1}$  and  $(z - \frac{1}{i\hbar}uv)_{-*}^{-1}$  are defined in the  $\kappa$ -ordering expression for  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ .*

Note that  $(z + \frac{1}{i\hbar}uv)_{+*}^{-1} * (-z - \frac{1}{i\hbar}uv)_{-*}^{-1}$  diverges for any ordering expression. However, the standard resolvent formula gives the following:

**Proposition 4.4** *If  $z + w \neq 0$ , then*

$$\frac{1}{z + w} \left( (z + \frac{1}{i\hbar}uv)_{+*}^{-1} + (w - \frac{1}{i\hbar}uv)_{-*}^{-1} \right)$$

*is an inverse of  $(z + \frac{1}{i\hbar}uv) * (w - \frac{1}{i\hbar}uv)$ . In particular, for every positive integer  $n$ , and for every complex number  $z$  such that  $\operatorname{Re} z > -(n + \frac{1}{2})$ ,*

$$\frac{1}{2n} \left( (1 + \frac{1}{n}(z + \frac{1}{i\hbar}uv))_{+*}^{-1} + (1 - \frac{1}{n}(z + \frac{1}{i\hbar}uv))_{-*}^{-1} \right)$$

*is an inverse of  $1 - \frac{1}{n^2}(z + \frac{1}{i\hbar}uv)_*^2$  in the  $\kappa$ -ordering expression for  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ .*

## 4.1 Analytic continuation of inverses

Recall that  $(z \pm \frac{1}{i\hbar} uv)^{-1}_{\pm*}$  are holomorphic on the domain  $\text{Re } z > -\frac{1}{2}$ . It is natural to expect that  $(z \pm \frac{1}{i\hbar} uv)^{-1}_{\pm*} = C(C(z \pm \frac{1}{i\hbar} uv))^{-1}_{\pm*}$  for any non-zero constant  $C$ . To confirm this, we set  $C = e^{i\theta}$  and consider the  $\theta$ -derivative

$$e^{i\theta} \int_{-\infty}^0 e_*^{e^{i\theta} t (z \pm \frac{1}{i\hbar} uv)} dt.$$

In the  $(\kappa, \tau)$ -ordering expression, the phase part of the integrand is bounded in  $t$  and the amplitude is given by

$$\frac{2e^{i\theta} tz}{(1-\kappa)e^{e^{i\theta} t/2} + (1+\kappa)e^{-e^{i\theta} t/2}}, \quad \kappa \neq 1.$$

Hence, the integral converges whenever  $\text{Re } e^{i\theta} (z \pm \frac{1}{2}) > 0$ , and by integration by parts this convergence does not depend on  $\theta$ . It follows that  $(z \pm \frac{1}{i\hbar} uv)^{-1}_{\pm*}$  are holomorphic on the domain  $\mathbb{C} - \{t; -\infty < t < -\frac{1}{2}\}$ .

Next, it is natural to expect that the bumping identity  $(uv)*v = v*(uv - i\hbar)$  gives the following “sliding identities”

$$v_+^{-1} * (z + \frac{1}{i\hbar} uv)^{-1}_{+*} * v = (z - 1 + \frac{1}{i\hbar} uv)^{-1}_{+*}, \quad v_+^{-1} * (z - \frac{1}{i\hbar} uv)^{-1}_{-*} * v = (z + 1 - \frac{1}{i\hbar} uv)^{-1}_{-*}$$

whenever one can use the inverse of  $v$  in a suitable ordering expression. In this section, analytic continuation will be produced via these sliding identities.

In this note, we state the sliding identity by using, instead of  $v^{-1}$ , the left inverse  $v^\circ$  of  $v$  given below. First of all, we remark that formula (10) also gives

$$(u*v)^{-1}_{-*} = -\frac{1}{i\hbar} \int_0^\infty e_*^{t \frac{1}{i\hbar} u*v} dt, \quad (v*u)^{-1}_{+*} = \frac{1}{i\hbar} \int_{-\infty}^0 e_*^{t \frac{1}{i\hbar} v*u} dt.$$

These gives left/right inverses of  $u, v$

$$v^\circ = u*(v*u)^{-1}_{+*}, \quad u^\bullet = v*(u*v)^{-1}_{-*},$$

for it is easy to see that

$$v*v^\circ = 1, \quad v^\circ*v = 1 - \varpi_{00}, \quad u*u^\bullet = 1, \quad u^\bullet*u = 1 - \varpi_{00}.$$

The bumping identity gives

$$v*(z + \frac{1}{i\hbar} uv)*v^\circ = z + 1 + \frac{1}{i\hbar} uv, \quad v^\circ*(z + \frac{1}{i\hbar} uv)*v = (1 - \varpi_{00})*(z - 1 + \frac{1}{i\hbar} uv).$$

The successive use of the bumping identity gives the following useful formula:

$$(24) \quad (u*(v*u)^{-1}_{+*})^n * \varpi_{00} = \frac{1}{n!} (\frac{1}{i\hbar} u)^n * \varpi_{00}.$$

Using  $v^\circ$  instead of  $v^{-1}$ , we can give the analytic continuation of inverses. However, we have to be careful about the continuity of the  $*$ -product. We compute

$$\begin{aligned}
v^\circ * (z + \frac{1}{i\hbar} uv)^{-1}_{+*} &= u * \int_{-\infty}^0 e_*^{t(\frac{1}{i\hbar} uv + \frac{1}{2})} dt * \int_{-\infty}^0 e_*^{s(z + \frac{1}{i\hbar} uv)} ds \\
&= u * \int_{-\infty}^0 \int_{-\infty}^0 e_*^{t(\frac{1}{i\hbar} uv + \frac{1}{2})} * e_*^{s(z + \frac{1}{i\hbar} uv)} dt ds \\
&= \int_{-\infty}^0 \int_{-\infty}^0 e^{t\frac{1}{2} + sz} u * e_*^{(t+s)\frac{1}{i\hbar} uv} dt ds \\
&= \int_{-\infty}^0 \int_{-\infty}^0 e^{t\frac{1}{2} + sz - (t+s)} e_*^{(t+s)\frac{1}{i\hbar} uv} * u dt ds.
\end{aligned}$$

Hence, we have the identity whenever both sides are defined:

$$\begin{aligned}
(v^\circ * (z + \frac{1}{i\hbar} uv)^{-1}_{+*}) * v &= \int_{-\infty}^0 \int_{-\infty}^0 e^{-t\frac{1}{2} + s(z-1)} e_*^{(t+s)\frac{1}{i\hbar} uv} * (u * v) dt ds \\
&= \int_{-\infty}^0 (u * v) * e_*^{t\frac{1}{i\hbar} uv} dt * \int_{-\infty}^0 e_*^{s(z-1 + \frac{1}{i\hbar} uv)} ds \\
&= (1 - \varpi_{00}) * (z - 1 + \frac{1}{i\hbar} uv)^{-1}_{+*}.
\end{aligned}$$

Remarking that

$$\varpi_{00} * (z - 1 + \frac{1}{i\hbar} uv)^{-1}_{+*} = (z - \frac{1}{2})^{-1} \varpi_{00},$$

whenever  $(z - 1 + \frac{1}{i\hbar} uv)^{-1}_{+*}$  is defined, we have

$$(25) \quad (v^\circ * (z + \frac{1}{i\hbar} uv)^{-1}_{+*}) * v + (z - \frac{1}{2})^{-1} \varpi_{00} = (z - 1 + \frac{1}{i\hbar} uv)^{-1}_{+*}.$$

Since  $(z - \frac{1}{2})^{-1} \varpi_{00}$  is always defined, we see that (25) gives the formula for analytic continuation. Using this, we have the following (see [12] and [14] for more details):

**Theorem 4.1** *The inverses  $(z + \frac{1}{i\hbar} uv)^{-1}_{+*}$ ,  $(z - \frac{1}{i\hbar} uv)^{-1}_{-*}$  extend to holomorphic functions in  $z$  on  $\mathbb{C} - \{-(\mathbb{N} + \frac{1}{2})\}$ . In particular,  $(z^2 - (\frac{1}{i\hbar} uv)^2)^{-1}_{\pm*}$  extend to holomorphic functions of  $z$  on this domain.*

The product  $(z + \frac{1}{i\hbar} uv)^{-1}_{+*} * (w + \frac{1}{i\hbar} uv)^{-1}_{+*}$  is naturally defined, but the formula in Theorem 4.1 looks strange at the first glance, because  $z + \frac{1}{i\hbar} uv$  is not zero at  $z = n + \frac{1}{2}$  and  $(z + \frac{1}{i\hbar} uv)^{-1}_{+*}$  is singular at  $z = n + \frac{1}{2}$ , but  $(z + \frac{1}{i\hbar} uv) * (z + \frac{1}{i\hbar} uv)^{-1}_{+*} = 1$  for  $z \notin -(\mathbb{N} + \frac{1}{2})$ .

Note that

$$\begin{aligned}
\int_{-\infty}^0 (z + \frac{1}{i\hbar} uv) * e_*^{t(z + \frac{1}{i\hbar} uv)} dt &= \begin{cases} 1 & \text{Re } z > -\frac{1}{2} \\ 1 - \varpi_{00} & z = -\frac{1}{2} \end{cases}, \\
\int_{-\infty}^0 (z - \frac{1}{i\hbar} uv) * e_*^{t(z - \frac{1}{i\hbar} uv)} dt &= \begin{cases} 1 & \text{Re } z > -\frac{1}{2} \\ 1 - \overline{\varpi}_{00} & z = -\frac{1}{2} \end{cases}.
\end{aligned}$$

As suggested by these formulas, we extend the definition of the  $*$ -product as follows: For every polynomial  $p(u, v)$  or  $p(u, v) = e_*^{s\frac{1}{i\hbar} uv}$ ,

$$(26) \quad p(u, v) * (z \pm \frac{1}{i\hbar} uv)^{-1}_{\pm*} = \lim_{N \rightarrow \infty} p(u, v) * \int_{-N}^0 e_*^{t(z \pm \frac{1}{i\hbar} uv)} dt.$$

Hence we have the formula

$$(27) \quad (z + \frac{1}{i\hbar} uv) * (z + \frac{1}{i\hbar} uv)^{-1}_{+*} = \begin{cases} 1 & \text{Re } z > -\frac{1}{2} \\ 1 - \varpi_{00} & z = -\frac{1}{2} \end{cases}.$$

Considering  $(v^\circ)^n * (z + \frac{1}{i\hbar} uv) * (z + \frac{1}{i\hbar} uv)^{-1}_{+*} * v^n = (v^\circ)^n * (z + \frac{1}{i\hbar} uv) * v^n * (v^\circ)^n * (z + \frac{1}{i\hbar} uv)^{-1}_{+*} * v^n$  and using the formula (24), we have the following:

**Theorem 4.2** *If we use definition (26) for the  $*$ -product, then*

$$(28) \quad (z + \frac{1}{i\hbar}uv) * (z + \frac{1}{i\hbar}uv)^{-1}_+ = \begin{cases} 1 & z \notin -(\mathbb{N} + \frac{1}{2}) \\ 1 - \frac{1}{n!} (\frac{1}{i\hbar}u)^n * \varpi_{00} * v^n & z = -(n + \frac{1}{2}) \end{cases},$$

$$(29) \quad (z - \frac{1}{i\hbar}uv) * (z - \frac{1}{i\hbar}uv)^{-1}_- = \begin{cases} 1 & z \notin -(\mathbb{N} + \frac{1}{2}) \\ 1 - \frac{1}{n!} (\frac{1}{i\hbar}v)^n * \overline{\varpi}_{00} * u^n & z = -(n + \frac{1}{2}) \end{cases}.$$

Although  $z = -(n + \frac{1}{2})$  are all removable singularities for (28) and (29) as a function of  $z$ , it is better to retain these singular points.

These formulas give in particular for every fixed positive integer  $m$

$$(30) \quad (1 + \frac{1}{m}(z + \frac{1}{i\hbar}uv)) * (1 + \frac{1}{m}(z + \frac{1}{i\hbar}uv))^{-1}_+ = \begin{cases} 1 & z \notin -(\mathbb{N} + m + \frac{1}{2}) \\ 1 - \frac{1}{k!} (\frac{1}{i\hbar}u)^k * \varpi_{00} * v^k & z = -(k + m + \frac{1}{2}) \end{cases}$$

for arbitrary  $k \in \mathbb{N}$ . We state the following identity for later use:

$$(31) \quad \varpi_{00} * v^n * (-n - \frac{1}{2} + \frac{1}{i\hbar}uv) = \varpi_{00} * (\frac{1}{i\hbar}u * v) * v^n = 0.$$

## 5 An infinite product formula

Recall the classical formula  $\sin \pi x = \pi x \prod_{k=1}^{\infty} (1 - \frac{x^2}{k^2})$ . Rewrite this as follows:

$$\prod_{k=1}^{\infty} (1 - \frac{x^2}{k^2}) = \frac{1}{2i} \int \chi_{[-\pi, \pi]}(t) e^{itx} dt = \lim_{n \rightarrow \infty} \int \prod_{k=1}^n (1 + \frac{1}{k^2} \partial_t^2) \delta(t) e^{itx} dt,$$

where  $\chi_{[-\pi, \pi]}(t)$  is the characteristic function of the interval  $[-\pi, \pi]$ . It follows that

$$\chi_{[-\pi, \pi]}(t) = 2i \lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + \frac{1}{k^2} \partial_t^2) \delta(t)$$

in the space of distributions.

For  $\kappa$  such that  $|\frac{\kappa+1}{\kappa-1}| \neq 1$ , so that  $:e_*^{it\frac{1}{i\hbar}uv}:_{\kappa}$  is not singular on  $t \in \mathbb{R}$ , we compute as follows:

$$\int \chi_{[-\pi, \pi]}(t) :e_*^{it(z \pm \frac{1}{i\hbar}uv)}:_{\kappa} dt = \int \chi_{[-\pi, \pi]}(t) e^{itz} :e_*^{\pm it\frac{1}{i\hbar}uv}:_{\kappa} dt.$$

Fixing a cut-off function  $\psi(t)$  of compact support such that  $\psi=1$  on  $[-\pi, \pi]$ , we see that

$$\int \chi_{[-\pi, \pi]}(t) :e_*^{t(z \pm \frac{1}{i\hbar}uv)}:_{\kappa} dt = 2i \lim_{n \rightarrow \infty} \int \prod_{k=1}^n (1 + \frac{1}{k^2} \partial_t^2) \delta(t) \psi(t) e^{tz} :e_*^{\pm it\frac{1}{i\hbar}uv}:_{\kappa} dt.$$

Integration by parts gives

$$\lim_{n \rightarrow \infty} \int \delta(t) \prod_{k=1}^n (1 + \frac{1}{k^2} \partial_t^2) \psi(t) e^{tz} :e_*^{\pm it\frac{1}{i\hbar}uv}:_{\kappa} dt = \lim_{n \rightarrow \infty} \prod_{k=1}^n : (1 + \frac{1}{k^2} \partial_t^2) e_*^{t(z \pm \frac{1}{i\hbar}uv)} :_{\kappa}.$$

Hence we have in the  $\kappa$ -ordering expression that

$$\int \chi_{[-\pi, \pi]}(t) :e_*^{it(z \pm \frac{1}{i\hbar}uv)}:_{\kappa} dt = 2i \lim_{n \rightarrow \infty} \prod_{k=1}^n (1 - \frac{1}{k^2} (z \pm \frac{1}{i\hbar}uv)^2)_{*}.$$

Noting that

$$\sin_* \pi(z \pm \frac{1}{i\hbar} uv) = \pi(z \pm \frac{1}{i\hbar} uv) * \int \chi_{[-\pi, \pi]}(t) e_*^{it(z \pm \frac{1}{i\hbar} uv)} dt \in \text{Hol}(\mathbb{C}^2),$$

we have

$$(32) \quad \sin_* \pi(z \pm \frac{1}{i\hbar} uv) = \pi(z \pm \frac{1}{i\hbar} uv) * \lim_{n \rightarrow \infty} * \prod_{k=1}^n (1 - \frac{1}{k^2} (z \pm \frac{1}{i\hbar} uv)^2)_*$$

in  $\text{Hol}(\mathbb{C}^2)$ . In particular, we have

**Proposition 5.1** *In the  $\kappa$ -ordering expression with  $|\frac{\kappa+1}{\kappa-1}| \neq 1$ , we have*

$$\sin_* \pi(z + \frac{1}{i\hbar} uv) = \pi(z + \frac{1}{i\hbar} uv) * \lim_{n \rightarrow \infty} \prod_{k=1}^n * (1 - \frac{1}{k^2} (z + \frac{1}{i\hbar} uv)^2).$$

This is identically zero on the set  $z \in \mathbb{Z} + \frac{1}{2}$ .

The formula in Proposition 5.1 may be rewritten as

$$\begin{aligned} \sin_* \pi(z + \frac{1}{i\hbar} uv) = \\ \pi(z + \frac{1}{i\hbar} uv) * \lim_{n \rightarrow \infty} \prod_{k=1}^n * (1 - \frac{1}{k^2} (z + \frac{1}{i\hbar} uv)) * e_*^{\frac{1}{k}(z + \frac{1}{i\hbar} uv)} * \prod_{k=1}^n * (1 + \frac{1}{k^2} (z + \frac{1}{i\hbar} uv)) * e_*^{-\frac{1}{k}(z + \frac{1}{i\hbar} uv)}. \end{aligned}$$

In §6, we will define a star gamma function via the two different inverses mentioned previously and give an infinite product formula for the star gamma function.

### 5.1 The product with $(z + \frac{1}{i\hbar} uv)^{-1}_{\pm*}$ and with $(1 + \frac{1}{m}(z + \frac{1}{i\hbar} uv))^{-1}_{+*}$

First we consider the product  $(z + \frac{1}{i\hbar} uv)^{-1}_{\pm*} * \sin_* \pi(z + \frac{1}{i\hbar} uv)$  in two different ways. One way is by defining:

$$(33) \quad \begin{aligned} & (z + \frac{1}{i\hbar} uv)^{-1}_{\pm*} * \sin_* \pi(z + \frac{1}{i\hbar} uv) \\ &= \lim_{n \rightarrow \infty} (z + \frac{1}{i\hbar} uv)^{-1}_{\pm*} * \left( (z + \frac{1}{i\hbar} uv) * \prod_{k=1}^n * (1 - \frac{1}{k^2} (z + \frac{1}{i\hbar} uv)^2) \right). \end{aligned}$$

Since  $(z + \frac{1}{i\hbar} uv) * \prod_{k=1}^n * (1 - \frac{1}{k^2} (z + \frac{1}{i\hbar} uv)^2)$  is a polynomial, Proposition 3.3, (28) and (31) give

$$(34) \quad (z + \frac{1}{i\hbar} uv)^{-1}_{\pm*} * \sin_* \pi(z \pm \frac{1}{i\hbar} uv) = \prod_{k=1}^{\infty} * (1 - \frac{1}{k^2} (z \pm \frac{1}{i\hbar} uv)^2).$$

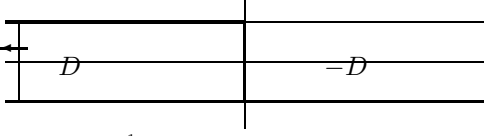
The second way is by defining

$$(35) \quad (z + \frac{1}{i\hbar} uv)^{-1}_{\pm*} * \sin_* \pi(z + \frac{1}{i\hbar} uv) = \lim_{N \rightarrow \infty} \int_{-N}^0 e_*^{t(z + \frac{1}{i\hbar} uv)} * \sin_* \pi(z + \frac{1}{i\hbar} uv).$$

This may be written as the complex integral

$$\frac{1}{2i} \int_{-\infty + \pi i}^{0 + \pi i} e_*^{t(z + \frac{1}{i\hbar} uv)} dt - \frac{1}{2i} \int_{-\infty - \pi i}^{0 - \pi i} e_*^{t(z + \frac{1}{i\hbar} uv)} dt.$$

If  $\operatorname{Re} z > -\frac{1}{2}$ , then adding  $-\frac{1}{2} \int_{-\pi}^{\pi} e^{it(z+\frac{1}{i\hbar}uv)} dt$  to this expressions gives the clockwise contour integral along the boundary of the domain  $D = \{z \in \mathbb{C}; \operatorname{Re} z < 0, -\pi < \operatorname{Im} z < \pi\}$ .



**Lemma 5.1**  $:e_*^{z+\frac{1}{i\hbar}uv}:$  has at most one singular point in the domain  $D \cup (-D)$ . If  $\operatorname{Re} \kappa > 0$ , then there is no singular point in  $D$ .

**Proof**  $:e_*^{z+\frac{1}{i\hbar}uv}:$   $\kappa = \frac{2}{(1-\kappa)e^{\frac{\pi}{2}} + (1+\kappa)e^{-\frac{\pi}{2}}} \exp \frac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{(1-\kappa)e^{\frac{\pi}{2}} + (1+\kappa)e^{-\frac{\pi}{2}}} \frac{2}{i\hbar} uv$ . Thus, the singular points are given by  $(1-\kappa)e^{\frac{\pi}{2}} + (1+\kappa)e^{-\frac{\pi}{2}} = 0$ . This gives  $e^z = \frac{\kappa+1}{\kappa-1}$ . If  $\kappa \neq \pm 1$ , then  $z = \log \frac{\kappa+1}{\kappa-1} + 2\pi ni$ . Thus, the domain  $D \cup (-D)$  contains at most one singular point.

If  $\operatorname{Re} \kappa > 0$ , then  $|\frac{\kappa+1}{\kappa-1}| > 1$  and the singular point (if it exists)  $z = \log \frac{\kappa+1}{\kappa-1} + 2\pi ni$  has a positive real part.  $\square$

**Proposition 5.2** Suppose  $\operatorname{Re} \kappa > 0$  and  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ . Then for  $z$  such that  $\operatorname{Re} z > -\frac{1}{2}$ , we have in the  $\kappa$ -ordering expression that

$$\lim_{N \rightarrow \infty} \int_{-N}^0 e_*^{t(z+\frac{1}{i\hbar}uv)} \sin_* \pi(z+\frac{1}{i\hbar}uv) = \frac{1}{2} \int_{-\pi}^{\pi} e_*^{it(z+\frac{1}{i\hbar}uv)} dt.$$

By (32) this integration gives the same result as (34), namely  $\prod_1^\infty (1 - \frac{1}{k^2}(z+\frac{1}{i\hbar}uv)^2)$ .

By the analytic continuation using  $v^\circ, v$  as before, we have the following:

**Proposition 5.3** Suppose  $\operatorname{Re} \kappa > 0$  and  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ . Then in the  $\kappa$ -ordering expression, the product  $\sin_* \pi(z+\frac{1}{i\hbar}uv) * (z+\frac{1}{i\hbar}uv)^{-1}_{+*}$  is an entire function of  $z$ . Namely, all singularities of  $(z+\frac{1}{i\hbar}uv)^{-1}_{+*}$  at  $-(N+\frac{1}{2})$  are cancelled out in formulas (30) and (31).

By a proof similar to that of Proposition 5.3, we obtain

**Proposition 5.4** Suppose  $\operatorname{Re} \kappa > 0$ , and  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$  Then in the  $\kappa$ -ordering expression,

$$\sin_* \pi(z - \frac{1}{i\hbar}uv) * (z - \frac{1}{i\hbar}uv)^{-1}_{-*}$$

is a well defined entire function of  $z$ .

In particular,  $\sin_* \pi(z+\frac{1}{i\hbar}uv) * (z^2 - (\frac{1}{i\hbar}uv)^2)^{-1}_{\pm*}$  is a holomorphic function of  $z$  in  $\mathbb{C}$ .

Consider next the product  $(1+\frac{1}{m}(z+\frac{1}{i\hbar}uv))^{-1}_{+*} \sin_* \pi(z+\frac{1}{i\hbar}uv)$ . Since

$$(1+\frac{1}{m}(z+\frac{1}{i\hbar}uv))^{-1}_{+*} = m(m+z+\frac{1}{i\hbar}uv)^{-1}_{+*},$$

and  $\sin_* \pi(z+m+\frac{1}{i\hbar}uv) = (-1)^m \sin_* \pi(z+\frac{1}{i\hbar}uv)$  by the exponential law, the product formula is essentially the same as above. Hence we see the following:

**Proposition 5.5** Suppose  $\operatorname{Re} \kappa > 0$ , and  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ . Then in the  $\kappa$ -ordering expression, the product  $\sin_* \pi(z+\frac{1}{i\hbar}uv) * (1+\frac{1}{m}(z+\frac{1}{i\hbar}uv))^{-1}_{+*}$  is an entire function of  $z$  with no removable singularity.

**Remark 2** Suppose  $\operatorname{Re} \kappa < 0$ , and  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ . Then the residue of  $e_*^{t(z+\frac{1}{i\hbar}uv)}$  at the singular point  $t = \log \frac{\kappa+1}{\kappa-1} + 2\pi ni$  in  $D$  gives the difference between the twosides of the equality in Proposition 5.3.

This observation shows that continuity does not hold for  $\kappa$ -ordering expressions.

Lemma 5.1 and formula (12) show that the integral  $\frac{1}{2\pi i} \int_{\partial D} e_*^{t(z+\frac{1}{i\hbar}uv)} dt$  gives the residue at the singular point in  $D$ . This residue will be computed in the last section.

## 6 Star gamma functions

We first recall the ordinary gamma function and beta function:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Substituting  $t = e^s$  gives

$$\Gamma(z) = \int_{-\infty}^\infty e^{-e^s} e^{sz} ds, \quad B(x, y) = \int_{-\infty}^0 e^{sx} (1-e^s)^{y-1} ds.$$

The star gamma function and the star beta function may be defined by replacing  $x$  with  $z \pm \frac{uv}{i\hbar}$ :

$$(36) \quad \begin{aligned} \Gamma_*(z \pm \frac{uv}{i\hbar}) &= \int_{-\infty}^\infty e^{-e^\tau} e_*^{\tau(z \pm \frac{uv}{i\hbar})} d\tau, \\ B_*(z \pm \frac{uv}{i\hbar}, y) &= \int_{-\infty}^0 e_*^{\tau(z \pm \frac{uv}{i\hbar})} (1-e^\tau)^{y-1} d\tau. \end{aligned}$$

The Weyl ordering expressions of these orderings are

$$\begin{aligned} : \Gamma_*(z \pm \frac{uv}{i\hbar}) :_0 &= \int_{-\infty}^\infty \frac{e^{-e^\tau + z\tau}}{\cosh \frac{1}{2}\tau} e^{\pm \frac{1}{i\hbar} uv \tanh \frac{1}{2}\tau} d\tau, \\ : B_*(z \pm \frac{uv}{i\hbar}, y) :_0 &= \int_{-\infty}^0 \frac{(1-e^\tau)^{y-1} e^{\tau z}}{\cosh \frac{1}{2}\tau} e^{\pm \frac{1}{i\hbar} uv \tanh \frac{1}{2}\tau} d\tau. \end{aligned}$$

The  $\kappa$ -ordering expressions are obtained by applying the intertwiner  $I_0^\kappa$  for  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ .

$$(37) \quad : \Gamma_*(z \pm \frac{uv}{i\hbar}) :_\kappa = \lim_{N, N' \rightarrow \infty} \int_{-N}^{N'} \frac{e^{-e^\tau + z\tau}}{\cosh \frac{1}{2}\tau} I_0^\kappa(e^{\pm \frac{1}{i\hbar} uv \tanh \frac{1}{2}\tau}) d\tau.$$

The right hand side converges on a dense open domain of  $\kappa$ .

**Proposition 6.1** *For every  $uv \in \mathbb{C}$ , and for every  $z \in \mathbb{C}$  such that  $\operatorname{Re} z > -\frac{1}{2}$ , the right hand side of (37) converges and is holomorphic with respect to  $z$ . However,  $\Gamma_*(-\frac{1}{2} \pm \frac{uv}{i\hbar})$  is singular.*

Throughout this section, ordering expressions are always restricted to  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ .

### 6.1 Analytic continuation of $\Gamma_*(z \pm \frac{uv}{i\hbar})$

As with the usual gamma function, integration by parts gives the identity

$$(38) \quad \Gamma_*(z+1 \pm \frac{uv}{i\hbar}) = (z \pm \frac{uv}{i\hbar}) * \Gamma_*(z \pm \frac{uv}{i\hbar}).$$

Using

$$\Gamma_*(z \pm \frac{uv}{i\hbar}) = (z \pm \frac{uv}{i\hbar})_{\pm}^{-1} * \Gamma_*(z+1 \pm \frac{uv}{i\hbar}),$$

and careful treating continuity insures the following

**Proposition 6.2**  $\Gamma_*(z \pm \frac{uv}{i\hbar})$  extends to a holomorphic function on  $z \in \mathbb{C} - \{-(\mathbb{N} + \frac{1}{2})\}$ .



Since  $e_*^{\tau(z \pm \frac{uv}{hi})} * \varpi_{00} = (z \pm \frac{1}{2})^{-1} \varpi_{00}$ , we see the following remarkable feature of these star functions

$$(39) \quad \begin{aligned} \Gamma_*(z \pm \frac{uv}{hi}) * \varpi_{00} &\equiv \lim_{N \rightarrow \infty} \int_{-N}^N e^{-e^\tau} e_*^{\tau(z \pm \frac{uv}{hi})} d\tau * \varpi_{00} = \Gamma(z \pm \frac{1}{2}) \varpi_{00} \\ B_*(z \pm \frac{uv}{hi}, y) * \varpi_{00} &\equiv \lim_{N \rightarrow \infty} \int_{-N}^N e_*^{\tau(z \pm \frac{uv}{hi})} (1 - e^\tau)^{y-1} d\tau * \varpi_{00} = B(z \pm \frac{1}{2}, y) \varpi_{00} \end{aligned}$$

## 6.2 An infinite product formula

We see in the same notation as above

$$(40) \quad B_*(z \pm \frac{uv}{hi}, 1) = \int_{-\infty}^0 e_*^{\tau(z \pm \frac{uv}{hi})} d\tau = (z + \frac{uv}{ih})_{*\pm}^{-1}, \quad \text{Re } z > -\frac{1}{2}.$$

We now compute

$$\Gamma_*(z \pm \frac{uv}{hi}) \Gamma(y) = \iint_{\mathbb{R}^2} e_*^{\tau(z \pm \frac{uv}{hi})} e^{\sigma y} e^{-(e^\tau + e^\sigma)} d\tau d\sigma.$$

We change variables by setting

$$\tau = t + s, \quad e^\sigma = e^t(1 - e^s), \quad \text{where } -\infty < t < \infty, \quad -\infty < s < 0.$$

Since  $e^\tau + e^\sigma = e^t$ , this gives a diffeomorphism of  $\mathbb{R} \times \mathbb{R}_-$  onto  $\mathbb{R}^2$ . The Jacobian is given by  $d\tau d\sigma = \frac{1}{1 - e^s} dt ds$ . Hence we have the fundamental relation between the gamma function and the beta function

$$(41) \quad \begin{aligned} \Gamma_*(z \pm \frac{uv}{hi}) \Gamma(y) &= \int_{-\infty}^0 \int_{-\infty}^0 e_*^{t(y + z \pm \frac{uv}{hi})} e^{-e^t} e_*^{s(z \pm \frac{uv}{hi})} (1 - e^s)^{y-1} dt ds \\ &= \Gamma_*(y + z \pm \frac{uv}{hi}) * B_*(z \pm \frac{uv}{hi}, y). \end{aligned}$$

Integration by parts gives

$$(z \pm \frac{uv}{hi}) * B_*(z \pm \frac{uv}{hi}, y + 1) = y B_*(1 + z \pm \frac{uv}{hi}, y + 1).$$

To prove this, note that

$$\begin{aligned} \frac{d}{d\tau} e_*^{\tau(z \pm \frac{uv}{hi})} &= (z \pm \frac{uv}{hi}) * e_*^{\tau(z \pm \frac{uv}{hi})}, \quad \frac{d}{d\tau} e^{-e^\tau} = -e^\tau e^{-e^\tau}, \\ \lim_{\tau \rightarrow \pm\infty} e^{-e^\tau + z\tau} e_*^{\pm\tau \frac{uv}{hi}} &= 0 \quad \text{for } \text{Re } z > -\frac{1}{2}. \end{aligned}$$

Since  $B_*(z \pm \frac{uv}{hi}, y + 1) = B_*(z \pm \frac{uv}{hi}, y) - B(1 + z \pm \frac{uv}{hi}, y)$ , we have the functional equation

$$(42) \quad B_*(z \pm \frac{uv}{hi}, y) = \frac{y + z \pm \frac{uv}{hi}}{y} * B_*(z \pm \frac{uv}{hi}, y + 1).$$

Iterate (42) to obtain

$$B_*(z \pm \frac{uv}{hi}, y) = \frac{(y + z \pm \frac{uv}{hi}) * (y + 1 + z \pm \frac{uv}{hi})}{y(y + 1)} * B_*(z \pm \frac{uv}{hi}, y + 2).$$

Using the notation

$$(a)_n = a(a + 1) \cdots (a + n - 1), \quad \{A\}_{*n} = A * (A + 1) * \cdots * (A + n - 1),$$

we have

$$(43) \quad B_*(z \pm \frac{uv}{\hbar i}, y) = \frac{\{y+z \pm \frac{uv}{\hbar i}\}_{*n}}{(y)_n} * B_*(z \pm \frac{uv}{\hbar i}, y+n).$$

Similarly, integration by parts gives the formula

$$(44) \quad \Gamma_*(1+z \pm \frac{uv}{\hbar i}) = (z \pm \frac{uv}{\hbar i}) * \Gamma_*(z \pm \frac{uv}{\hbar i}), \quad \text{for } \operatorname{Re} z > -\frac{1}{2}.$$

Iterate (44) to obtain

$$(45) \quad \Gamma_*(n+1+z \pm \frac{uv}{\hbar i}) = \Gamma_*(z \pm \frac{uv}{\hbar i}) * \{z \pm \frac{uv}{\hbar i}\}_{*n}.$$

**Lemma 6.1**  $B_*(z \pm \frac{uv}{\hbar i}, n+1) = n! \prod_{k=0}^n (k+z \pm \frac{uv}{\hbar i})_{\pm*}^{-1}.$

**Proof** The right hand side of the above equality will be denoted by  $\frac{n!}{\{z \pm \frac{uv}{\hbar i}\}_{*n+1}^{(\pm)}}$ .

The case  $n = 0$  is given by (40). Suppose the formula holds for  $n$ . For the case  $n+1$ , we see that

$$B_*(z \pm \frac{uv}{\hbar i}, n+2) = \int_{-\infty}^0 e_*^{\tau(z \pm \frac{uv}{\hbar i})} (1-e^\tau)(1-e^\tau)^n d\tau = \frac{n!}{\{z \pm \frac{uv}{\hbar i}\}_{*n+1}^{(\pm)}} - \frac{n!}{\{1+z \pm \frac{uv}{\hbar i}\}_{*n+1}^{(\pm)}}.$$

It follows that

$$B_*(z \pm \frac{uv}{\hbar i}, n+2) = \frac{(n+1)!}{\{z \pm \frac{uv}{\hbar i}\}_{*n+2}^{(\pm)}}.$$

□

In this subsection, we give an infinite product formula for the  $*$ -gamma function. By Lemma 6.1, we see that

$$\int_{-\infty}^0 e_*^{\tau(z \pm \frac{uv}{\hbar i})} (1-e^\tau)^n d\tau = \frac{n!}{\{z \pm \frac{uv}{\hbar i}\}_{*n+1}^{(\pm)}}, \quad \operatorname{Re} z > -\frac{1}{2}.$$

Replacing  $e^\tau$  by  $\frac{1}{n}e^{\tau'}$ , namely setting  $\tau = \tau' - \log n$  in the left hand side, and multiplying both side by  $e_*^{(\log n)(z \pm \frac{uv}{\hbar i})}$ , we have

$$(46) \quad \int_{-\infty}^{\log n} e_*^{\tau'(z \pm \frac{uv}{\hbar i})} (1-\frac{1}{n}e^{\tau'})^n d\tau' = \frac{n!}{\{z \pm \frac{uv}{\hbar i}\}_{*n+1}^{(\pm)}} * e_*^{(\log n)(z \pm \frac{uv}{\hbar i})}.$$

**Lemma 6.2** *The Weyl ordering expression of the left hand side of (46) converges as  $n \rightarrow \infty$  to  $\int_{-\infty}^{\infty} e_*^{\tau'(z \pm \frac{uv}{\hbar i})} e^{-e^{\tau'}} d\tau'$  in  $\operatorname{Hol}(\mathbb{C}^2)$ .*

**Proof** Obviously,  $\lim_{n \rightarrow \infty} (1-\frac{1}{n}e^{\tau'})^n = e^{-e^{\tau'}}$  uniformly on each compact subset as a function of  $\tau'$ . In the Weyl ordering expression, it is easy to show that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\log n} e_*^{\tau'(z \pm \frac{uv}{\hbar i})} e^{-e^{\tau'}} d\tau' = \int_{-\infty}^{\infty} e_*^{\tau'(z \pm \frac{uv}{\hbar i})} e^{-e^{\tau'}} d\tau'$$

in  $\operatorname{Hol}(\mathbb{C}^2)$ . Thus it is enough to show that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\log n} e_*^{\tau'(z \pm \frac{uv}{\hbar i})} (e^{-e^{\tau'}} - (1-\frac{1}{n}e^{\tau'})^n) d\tau' = 0$$

in  $Hol(\mathbb{C}^2)$ . This is easy in the Weyl ordering. Applying the intertwiner gives the desired result.  $\square$

The right hand side of (46) equals

$$e_*^{(\log n - (1 + \frac{1}{2} + \dots + \frac{1}{n}))(z \pm \frac{uv}{\hbar i})} * (z \pm \frac{uv}{\hbar i})_{\pm *}^{-1} * \prod_{k=1}^n \left( \left( 1 + \frac{z \pm \frac{uv}{\hbar i}}{k} \right)_{\pm *}^{-1} * e_*^{\frac{z \pm \frac{uv}{\hbar i}}{k}} \right), \quad \operatorname{Re} z > -\frac{1}{2}.$$

The left hand side converges, and  $\lim_{n \rightarrow \infty} e_*^{(\log n - (1 + \frac{1}{2} + \dots + \frac{1}{n}))(z \pm \frac{uv}{\hbar i})} = e_*^{-\gamma(z \pm \frac{uv}{\hbar i})}$  obviously, where  $\gamma$  is Euler's constant. By the continuity of the  $*$ -multiplication  $e_*^{\frac{uv}{\hbar i}} *$ , we have the convergence in  $Hol(\mathbb{C}^2)$  of

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n * \left( \left( 1 + \frac{1}{k} \left( z \pm \frac{uv}{\hbar i} \right) \right)_{\pm *}^{-1} * e_*^{\frac{1}{k} \left( z \pm \frac{uv}{\hbar i} \right)} \right).$$

Hence we have the convergence in  $Hol(\mathbb{C}^2)$  of the infinite product formula

$$(47) \quad \Gamma_*(z + \frac{uv}{\hbar i}) = e_*^{-\gamma(z + \frac{uv}{\hbar i})} * (z + \frac{uv}{\hbar i})_{+ *}^{-1} * \prod_{k=1}^{\infty} * \left( \left( 1 + \frac{1}{k} \left( z + \frac{1}{i\hbar} uv \right) \right)_{+ *}^{-1} * e_*^{\frac{1}{k} \left( z + \frac{1}{i\hbar} uv \right)} \right)$$

Fix  $m \in \mathbb{N}$ . Multiplying  $(1 + \frac{1}{m}(z + \frac{uv}{\hbar i}))e_*^{-\frac{1}{m}(z + \frac{uv}{\hbar i})}$  to both side of (47) and using the abbreviated notation

$$\prod_{k \neq m} (z = a) = (z + \frac{uv}{\hbar i})_{+ *}^{-1} \prod_{k \neq m} * \left( \left( 1 + \frac{1}{k} \left( a + \frac{1}{i\hbar} uv \right) \right)_{+ *}^{-1} * e_*^{\frac{1}{k} \left( a + \frac{1}{i\hbar} uv \right)} \right)$$

we have

$$(48) \quad \begin{aligned} & \left( 1 + \frac{1}{m} \left( z + \frac{uv}{i\hbar} \right) \right) * e_*^{-\frac{1}{m} \left( z + \frac{uv}{\hbar i} \right)} * \Gamma_*(z + \frac{uv}{\hbar i}) \\ &= \begin{cases} \prod_{k \neq m} (z = z) & z \notin -(\mathbb{N} + m + \frac{1}{2}) \\ \prod_{k \neq m} (z = -n - m - \frac{1}{2}) * \left( 1 - \frac{1}{n!} \left( \frac{1}{i\hbar} u \right)^n * \varpi_{00} * v^n \right) & z = -(n + m + \frac{1}{2}) \end{cases} \end{aligned}$$

where  $n \in \mathbb{N}$ . As opposited to the case that  $(1 + \frac{1}{m}(z + \frac{uv}{i\hbar}))_{+ *}^{-1} * \sin_* \pi(z + \frac{1}{i\hbar} uv)$  is entire function (cf. Proposition 5.5), there are removable singularities with respect to  $z$ .

Multiplying  $\prod_{k=1}^{\infty} (1 + \frac{1}{k}(z + \frac{uv}{\hbar i}))e_*^{-\frac{1}{k}(z + \frac{uv}{\hbar i})}$  to both sides of (47) and using (48), we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \prod_{k=1}^N * \left( \left( 1 + \frac{1}{k} \left( z + \frac{1}{i\hbar} uv \right) \right) * e_*^{-\frac{1}{k} \left( z + \frac{1}{i\hbar} uv \right)} \right) * \Gamma_*(z + \frac{1}{i\hbar} uv) \\ &= \begin{cases} 1 & z \notin -(\mathbb{N} + \frac{1}{2}) \\ 1 - \sum_{k=0}^n \frac{1}{k!} \left( \frac{1}{i\hbar} u \right)^k * \varpi_{00} * v^k, & z = -(n + \frac{1}{2}), \end{cases} \end{aligned}$$

in  $Hol(\mathbb{C}^2)$ , where  $n \in \mathbb{N}$ .

## 7 Products with $\sin_* \pi(z + \frac{1}{i\hbar} uv)$

In this section we show that  $\sin_* \pi(z + \frac{1}{i\hbar} uv) * \Gamma_*(z + \frac{1}{i\hbar} uv)$  is well defined as an entire function of  $z$ . By recalling Euler's reflection formula, this product may be understood as

$\frac{1}{\Gamma_*(1-(z+\frac{1}{i\hbar}uv))}$ . First, for  $\text{Re } z > -\frac{1}{2}$ , we define the product by the integral

$$(49) \quad \begin{aligned} & 2i \sin_* \pi(z + \frac{1}{i\hbar}uv) * \Gamma_*(z + \frac{1}{i\hbar}uv) \\ &= \lim_{T, T' \rightarrow \infty} \int_{-T}^{T'} (e_*^{\pi i(z + \frac{1}{i\hbar}uv)} - e_*^{-\pi i(z + \frac{1}{i\hbar}uv)}) * e^{-e^\tau} e_*^{\tau(z + \frac{uv}{i\hbar})} d\tau \\ &= \int_{-\infty}^{\infty} e^{-e^\tau} (e_*^{(\tau + \pi i)(z + \frac{uv}{i\hbar})} - e_*^{(\tau - \pi i)(z + \frac{uv}{i\hbar})}) d\tau. \end{aligned}$$

The  $\kappa$ -ordering expression of (49) is given as follows:

$$:(49):_\kappa = \int_{-\infty + \pi i}^{\infty + \pi i} e^{-e^{\tau - \pi i}} e_*^{\tau(z + \frac{uv}{i\hbar})} d\tau - \int_{-\infty - \pi i}^{\infty - \pi i} e^{-e^{\tau + \pi i}} e_*^{\tau(z + \frac{uv}{i\hbar})} d\tau.$$

By using  $e^{-e^{\tau - \pi i}} = e^{-e^{\tau + \pi i}}$ , this is given by the integral

$$(\int_{-\infty + \pi i}^{\infty + \pi i} - \int_{-\infty - \pi i}^{\infty - \pi i}) e^{e^\tau} e_*^{\tau(z + \frac{uv}{i\hbar})} d\tau.$$

Note this is not a contour integral, but it is defined for  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ .

After this procedure, we use the analytic continuation via (32), (38) to obtain the following, which is our main result:

**Theorem 7.1**  $\sin_* \pi(z + \frac{1}{i\hbar}uv) * \Gamma_*(z + \frac{1}{i\hbar}uv)$  is defined as an entire function of  $z$ , vanishing at  $z \in \mathbb{N} + \frac{1}{2}$  in any  $\kappa$ -ordering expression such that  $\text{Re } \kappa < 0$ , and  $\kappa \in \mathbb{C} - \{\kappa \geq 1\} \cup \{\kappa \leq -1\}$ .

After careful argument about associativity, (49) can be expressed as an infinite product

$$(50) \quad \sin_* \pi(z + \frac{1}{i\hbar}uv) * \Gamma_*(z + \frac{1}{i\hbar}uv) = \prod_{k=1}^{\infty} * \left( (1 - \frac{1}{k}(z + \frac{uv}{i\hbar})) * e_*^{\frac{1}{k}(z + \frac{uv}{i\hbar})} \right).$$

Recalling the reflection formula, we may define

$$\frac{1}{\Gamma_*}(1 - (z + \frac{1}{i\hbar}uv)) = \sin_* \pi(z + \frac{1}{i\hbar}uv) * \Gamma_*(z + \frac{1}{i\hbar}uv).$$

By this we see that

$$\frac{1}{\Gamma_*}(1 - (z + \frac{1}{i\hbar}uv)) \Big|_{z=\frac{1}{2}} = 0.$$

This supports the interpretation that  $(\frac{1}{2} + \frac{1}{i\hbar}uv)$  is an indeterminate living in the set of positive integers  $\mathbb{N} = \{1, 2, 3, \dots\}$ .

We can form the product  $\frac{1}{\Gamma_*}(1 - (z + \frac{1}{i\hbar}uv)) * (1 - \frac{1}{n}(z + \frac{1}{i\hbar}uv))_{-*}^{-1} * e_*^{-\frac{1}{n}(z + \frac{1}{i\hbar}uv)}$ . At first glance, this looks like

$$\prod_{k \neq n} * (1 - \frac{1}{k}(z + \frac{1}{i\hbar}uv)) * e_*^{\frac{1}{k}(z + \frac{1}{i\hbar}uv)}$$

and hence as an entire function with respect to  $z$ .

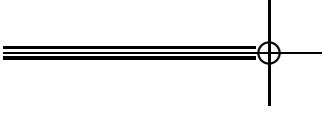
However, note that  $(1 - \frac{1}{n}(z + \frac{1}{i\hbar}uv))_{-*}^{-1}$  is singular at  $n - z \in -\mathbb{N} - \frac{1}{2}$ , i.e.  $z \in k + \frac{1}{2}$  for  $k \geq -n$ , and the same calculation as in (48) shows that

$$\sin_* \pi(z + \frac{1}{i\hbar}uv) * (\Gamma_*(z + \frac{1}{i\hbar}uv) * (1 - \frac{1}{n}(z + \frac{1}{i\hbar}uv))_{-*}^{-1})$$

is not defined as an entire function, since some matrix elements appear in the formula as removable singularities. Some additional arguments is may be requested, since these are all removable singularities in the usual calculation.

## 7.1 Additional support for the discrete interpretation

We give another formula to support the discrete interpretation for  $\frac{1}{\hbar i}uv$ . Recall Hankel's formula



$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_C e^t t^{-s} dt, \quad (\text{cf. [19] p. 244})$$

where  $C$  is taken to be a line from  $-\infty$  to  $-\delta$ , then a circle of radius  $\delta$  in the positive direction, and finally a line from  $-\delta$  to  $-\infty$ .

Setting  $s = \frac{1}{2} - \frac{1}{\hbar i}uv = -\frac{1}{\hbar i}u^*v$ , we want to prove  $\int_C e^t t_*^{\frac{1}{\hbar i}u^*v} dt = 0$  as additional support for the discrete interpretation.

By setting  $t = e^{\tau + \pi i}$ , it is easy to see that the Weyl ordering expression of this integral is equal to

$$\begin{aligned} : \int_{-\infty}^0 e^t t_*^{\frac{1}{\hbar i}uv} dt :_0 &= : \int_{-\infty}^{\infty} e^{e^{\tau + \pi i}} e_*^{(\tau + \pi i)(1 + \frac{1}{\hbar i}uv)} d\tau :_0 \\ &= \int_{-\infty}^{\infty} e^{e^{\tau + \pi i}} \frac{e^{\tau + \pi i}}{\cosh(\tau + \pi i)} e^{\frac{1}{\hbar i}uv \tanh(\tau + \pi i)} d\tau. \end{aligned}$$

Hence the integral

$$: \int_{-\infty}^0 e^t t_*^{\frac{1}{\hbar i}uv} dt :_0 = \int_{-\infty}^{\infty} e^{-e^{\tau}} \frac{e^{\tau}}{\cosh(\tau)} e^{\frac{1}{\hbar i}uv \tanh(\tau)} d\tau$$

exists and our integral vanishes on the axis part of  $C$ . Thus, setting  $t = e^{\tau} e^{i\theta}$ , we consider for a fixed real  $\tau \ll 0$

$$: A_*\left(\frac{uv}{\hbar i}\right) :_0 = \frac{1}{2\pi} \int_0^{2\pi} : e^{e^{\tau} e^{i\theta} + (\tau + i\theta)} e_*^{(\tau + i\theta)(\frac{1}{\hbar i}uv)} :_0 d\theta.$$

This can be written as

$$\frac{1}{2\pi} \int_0^{2\pi} e^{e^{\tau} e^{i\theta}} \frac{e^{\tau + i\theta}}{\cosh(\tau + i\theta)} e^{\frac{1}{\hbar i}uv \tanh(\tau + i\theta)} d\theta.$$

We easily see the following

$$\textbf{Lemma 7.1} \quad \lim_{\tau \rightarrow -\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{e^{\tau} e^{i\theta} + (\tau + i\theta)} e_*^{(\tau + i\theta)(\frac{2}{\hbar i}uv)} d\theta = 0.$$

Lemma 7.1 suggests that we write  $\frac{1}{\Gamma_*}\left(z + \frac{u^*v}{\hbar i}\right) \Big|_{z=0} = 0$ , although this is not rigorous.

## 7.2 The residue of $e_*^{t(z + \frac{1}{\hbar i}uv)}$

We first use the Weyl ordering expression. The  $\kappa$ -ordering expression is obtained via the intertwiner.

**Lemma 7.2** *Let  $C_k$  be a small circle of radius  $\frac{\pi}{4}$  with the center at  $\zeta = i\pi(k + \frac{1}{2})$ . Then the contour integral  $\frac{1}{2\pi i} \int_{C_k} : e_*^{\zeta(z + \frac{1}{\hbar i}2uv)} :_0 d\zeta$  gives the residue of  $e^t t_*^{\frac{1}{\hbar i}uv}$  and this is an entire function of  $X = (z, \frac{1}{\hbar i}2uv)$ .*

The continuity of the multiplication  $(z + \frac{1}{i\hbar}2uv)*$  requires that this function must satisfy the equation

$$(51) \quad (z + \frac{1}{i\hbar}2uv)*_0 \int_{C_k} :e_*^{\zeta(z + \frac{1}{i\hbar}2uv)}:_0 d\zeta = 0,$$

since (51) equals  $\int_{C_k} \frac{d}{d\zeta} e_*^{\zeta(z + \frac{1}{i\hbar}2uv)} d\zeta$ . For simplicity, we set

$$w = \frac{1}{\hbar}2uv, \quad \Phi_k(z, w) = \frac{1}{2\pi i} \int_{C_k} :e_*^{\zeta(z + \frac{1}{i\hbar}2uv)}:_0 d\zeta.$$

Equation (51) is  $(iz+w)*_0\Phi_k(z, w) = 0$ . Hence by the Moyal product formula,  $\Phi_k(z, w)$  must satisfy the equation

$$(52) \quad (iz+w)f(x) + f(w)' + wf(w)'' = 0,$$

independent of  $k$ . It is not difficult to see that equation (52) has the unique holomorphic solution  $f$  with initial condition  $f(0) = 1$ .

For  $f(w) = e^{aw}g(bw)$ , (52) can be rewritten as

$$b^2wg''(bw) + (2abw+b)g'(bw) + ((a^2+1)w+a+iz)g(bw) = 0.$$

Thus  $g(w)$  must satisfy the equation

$$(53) \quad wg''(w) + (1 + \frac{2a}{b}w)g'(w) + (\frac{a^2+1}{b^2}w + \frac{a+iz}{b})g(w) = 0.$$

Setting  $a = -\frac{1}{2}b = \pm i$ , we have a Laguerre equation

$$(54) \quad wg''(w) + (1-w)g'(w) + \frac{1}{2}(\mp z - 1)g(w) = 0,$$

where solution is known to be an entire function of exponential growth with respect to  $w$ .

Equation (54) gives two expressions for the solutions of (52) using the Laguerre functions  $L_\nu^{(0)}(2iw)$ :

$$\Psi_z(w) = e^{-iw}L_{\frac{1}{2}(z-1)}^{(0)}(2iw), \quad \Psi_z(w) = e^{iw}L_{-\frac{1}{2}(z+1)}^{(0)}(-2iw),$$

where

$$L_\nu^{(0)}(w) = \sum_{n=0}^{\infty} \frac{(-\nu)_n}{(n!)^2} w^n, \quad \nu = \frac{1}{2}(\mp z - 1).$$

Here we use the notation

$$(a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1.$$

By this observation, we see that  $\Phi_k(z, x) = c_k\Psi_z(x)$ , but the constant  $c_k$  is not fixed by this method. To fix the constant we remark that  $\Psi_z(x)$  is also analytic in the variable  $z$ . The constant  $c_k$  is fixed by investigating the case  $z = 0$ .

The residue of  $e_*^{t\frac{1}{i\hbar}uv}$  is obtained in the Weyl ordering by the contour integral

$$\int_{-\infty}^{\infty} (e_*^{(t-\pi i)\frac{1}{i\hbar}uv} - e_*^{(t+\pi i)\frac{1}{i\hbar}uv}) dt.$$

Since  $e_*^{(t-\pi i)\frac{1}{i\hbar}uv} = -e_*^{(t+\pi i)\frac{1}{i\hbar}uv}$ , this is given by (14).

**Lemma 7.3** *The residue of  $\frac{1}{\cosh \zeta} e^{(\frac{1}{i\hbar} \tanh \zeta)2uv}$  at  $\zeta = i\pi(k + \frac{1}{2})$  is*

$$(-1)^k (-i) \sqrt{2\pi} J_0\left(\frac{2}{\hbar}uv\right),$$

where  $J_0$  is the Bessel function with the eigenvalue 0.

Comparing these we know the residue of  $e_*^{t(z + \frac{1}{i\hbar}uv)}$ .

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